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Statistics of a filtered telegraph signal

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Abstract. Equations are derived which describe the time evolution of the probability density and corresponding characteristic function of a telegraph signal which has passed through a detuned Lorentzian filter. A closed form expression for the characteristic function is obtained for the tuned case and the predicted joint statistics and correlation properties are reviewed in the context of earlier results. Low-order correlation properties for the more general detuned case are calculated. It is shown that the stationary single interval statistics can be generated by a random phasor moving in a space of fractional dimensions and that a simple transform of variable leads to distributions which are stable in this space.

1. Introduction

A telegraph signal is a signal that switches randomly between two values. Telegraph signals are widely encountered throughout science and engineering. They have been used as simple binary random models for physical phenomena ranging from turbulent mixing and biological dispersal, to quantum jumps. Several recent applications of telegraph signal theory have been concerned with the propagation of radiation through dense scattering media and have sought to exploit properties of persistent or *correlated* random walk models. It is argued that the statistical properties of the transmitted radiation are governed by generalizations of the 'telegrapher's' equation governing the flow of electricity in cables, which was derived in a classic paper written by Goldstein in the early 1950s [1]. This equation provides a statistical description of the outcome of a correlated random walk and is parametrized by a characteristic time and velocity [2].

An earlier interest of one of the present authors in telegraph signals arose in the search for a simple non-Gaussian random surface-scattering model. The electromagnetic field scattered by a surface of rectangular grooves at normal incidence is, in a physical optics approximation, simply proportional to the integral of a telegraph signal (representing the surface profile) over the illuminated area of surface and its statistical properties are governed by the telegrapher's equation [3,4]. This type of model is relevant to the performance of infrared systems employing optical elements with diamond-turned surfaces.

Our more recent activity has been stimulated in part by the development of a diagnostic technique for semiconductor lasers based on electrical filtering after heterodyne detection [5]. According to one model for laser light, this form of post-detection processing could again lead to integration over a telegraph signal. However, in this case the integration is weighted by a memory function proportional to the Fourier transform of the applied filter shape rather than being over a fixed time interval or hard aperture as in the applications mentioned previously.

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In the case of a tuned *Lorentzian* filter it was found that exact analytical expressions for the equilibrium single-interval probability density and low-order coherence functions of the signal could be derived [6]. Some results were also obtained for the detuned case where the filter profile is not centred at zero frequency.

Since the publication of [6], a significant body of past literature concerning the Lorentzian filtered telegraph signal, stretching from the late 1950s to the early 1970s, has come to our attention. Early work was published by Wonham and Fuller [7, 8] and McFadden [9], and the subject was comprehensively reviewed in two papers by Pawula [10, 11] who has made several more recent contributions to the field [12–15]. An interesting feature of the mathematical analyses presented in these papers is the derivation of telegrapher-type equations with variable coefficients. This is a topic which has been considered recently in more general terms by Masoliver and Weiss [16] and relates to variations of the weighting or *velocity* applied to each step of the random walk by the filter. However, most recent work has continued to highlight solutions of the constant-velocity telegrapher's equation and to develop generalizations in more than one dimension for applications such as light propagation in turbid media [17–25].

Our aim in writing this paper is to extend the calculations of previous authors to include the effect of detuning, when the Lorenztian filter is not centred at zero frequency. In doing so we use a simple approach to the derivation of a Fokker–Planck equation which can readily be generalized to other types of dichotomous random walk. This enables us to review and simplify existing formulae as well as derive a number of new results.

The Lorentzian filtered random telegraph signal can be written as

$$E(t) = \lambda \int_{-\infty}^{t} \mathrm{d}t' \, T(t') \exp[\lambda(t' - t) + \mathrm{i}\omega(t' - t)]. \tag{1}$$

Here, the telegraph signal $T = \pm 1$ and its zero crossings are a random (Poisson) train of events with characteristic exponentially distributed inter-event time $1/\gamma$. The exponential memory function corresponds to a Lorentzian filter of linewidth λ centred at frequency ω. This uses the notation of [6] where we noted that (1) is also an expression for the complex wave amplitude in the far field of a randomly grooved phase object (or 'screen') illuminated by a coherent beam of radiation with a negative exponential intensity profile. New results obtained in the current analysis will be presented alongside statistical properties calculated by previous authors, providing a more comprehensive overview of the problem than has appeared hitherto, including generalization to the detuned case. We shall draw attention to a number of features of the results which have not been discussed in previous publications. These include an analogue of a so-called 'memory' effect which has featured prominently in the recent literature on speckle phenomena [26], and the close relationship between the single interval statistics derived here, the properties of an n-dimensional random phasor and the class of Levy-stable distributions [27].

In the next section equations for the characteristic function of the conditional probability density of a Lorentzian filtered telegraph signal will be derived from first principles, assuming that the filter is tuned (i.e. centred at zero frequency). In section 3 exact solutions for the characteristic functions of the conditional and joint probability densities will be presented for the tuned case, whilst in section 4 expressions for the single interval statistics and the low-order correlation properties will be given. Equations for the generalized case, included detuning are derived in section 5, and some low-order correlation properties for the detuned filter are presented. The results will be discussed in section 6, drawing attention to features of particular current interest. A summary and concluding remarks are given in section 7.

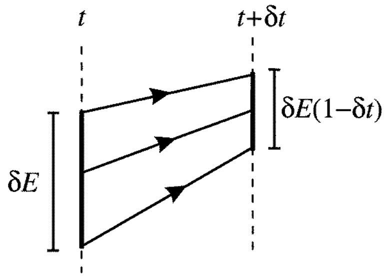

Figure 1. Diagram showing how the probability density increases with time as a result of different trajectories converging into a smaller interval (note E and t scales are not the same).

2. Derivation of equations for the conditional and joint statistics

In this section we will derive differential equations for the conditional probability density of E defined by formula (1). These equations have been given previously for the tuned case $\omega = 0$ [8, 10], the present derivation has the advantage of being easily extended to the case of a detuned filter.

Considering first the tuned case, equation (1) can be written in differential form as

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -E \pm 1 \tag{2}$$

where the ± 1 refers to the two different states of the telegraph signal T, and the time t has been normalized by the time constant of the filter λ. If we consider E as a point in a one-dimensional space, equation (2) defines a variable velocity which depends on the position E, as well as the state of the telegraph signal. Thus, motion of E can be considered as a random walk with variable velocity.

Following Wonham [8], we define two probability densities:

$$P^{+}(E, t)\,\mathrm{d}E \qquad P^{-}(E, t)\,\mathrm{d}E \tag{3}$$

where P^{+} is the probability that at time t the filtered signal lies between E and $E + \mathrm{d}E$, and the telegraph signal has value +1. P^{-} is the corresponding density for the -1 state of the telegraph signal.

Consider what happens during a small time interval δt. If δt is sufficiently small (such that terms in δt^2 can be neglected), there is a negligible probability of more than one change in the state of the telegraph signal during the interval. Thus the differential change in the probability during the interval can be expressed as a sum of two terms, one corresponding to there being no change in the state of the telegraph signal, and the other corresponding to a single transition during the interval.

If the density P^{+} is $P^{+}(E, t)\,\mathrm{d}E$ across an interval $\mathrm{d}E$ at time t, the density at time $t + \delta t$ will have changed even if there is no change in the state of the telegraph signal. This is because the variable velocity defined by (2) results in a compression of the interval over which P^{+} is defined. This is illustrated in figure 1.

The value of E lies between E and $E + \mathrm{d}E$ at time t. After time δt has elapsed, it is between $E + (1 - E)\,\delta t$ and $E + (1 - E)\,\delta t + \mathrm{d}E(1 - \delta t)$. Thus, if the telegraph signal remains in the +1 state the probability density after time δt is given by

$$\begin{aligned} &P^{+}(E + (1 - E)\,\delta t, t + \delta t)\,\mathrm{d}E(1 - \delta t) \\ &\qquad \approx \left(P^{+}(E, t)(1 - \delta t) + (1 - E)\frac{\partial P^{+}}{\partial E}\delta t + \frac{\partial P^{+}}{\partial t}\delta t \right) \mathrm{d}E. \end{aligned} \tag{4}$$

Here a Taylor expansion has been carried out and higher terms in δt have been neglected to give the expression on the right-hand side. Assuming that the state of the telegraph signal is determined by a sequence of Bernoulli trials, the probability of this outcome (i.e. the telegraph signal remaining in the +1 state) is $(1-\eta\delta t)$, where $\eta\delta t$ is the probability of a transition to the -1 state, and the normalized transition probability is $\eta = 2\gamma/\lambda$. Thus, in order to calculate the total probability for P^+, we need to equate (4) to $(1-\eta\delta t)P^+(E,t)\,\mathrm{d}E$ and add the second term, which is the probability of a transition from the -1 state to the +1 state occurring during δt. To first order in δt this second term is simply given by $\eta\delta t P^-(E,t)$. This results in the following equation:

$$(E-1)\frac{\partial P^+}{\partial E} - (\eta-1)P^+ + \eta P^- = \frac{\partial P^+}{\partial t}. \tag{5}$$

Similar considerations for P^- give a second equation

$$(E+1)\frac{\partial P^-}{\partial E} - (\eta-1)P^- + \eta P^+ = \frac{\partial P^-}{\partial t}. \tag{6}$$

These two equations were first given by Wonham in [8]. Similar arguments can be used to derive equations for the detuned case, which is discussed in section 5.

The derivation method given above can be extended to a more general random walk with a variable velocity which is an arbitrary function of E; this will be the subject of a future paper.

It is convenient to work with the characteristic function, which is the Fourier transform of the probability distribution. That is:

$$C^+(k,t) = \int_{-\infty}^{\infty} P^+(E,t)\mathrm{e}^{-\mathrm{i}kE}\mathrm{d}E. \tag{7}$$

Taking transforms of equations (5) and (6) gives

$$-k\frac{\partial C^+}{\partial k} - (\eta+\mathrm{i}k)C^+ + \eta C^- = \frac{\partial C^+}{\partial t} \tag{8}$$

$$-k\frac{\partial C^-}{\partial k} - (\eta+\mathrm{i}k)C^- + \eta C^+ = \frac{\partial C^-}{\partial t}. \tag{9}$$

3. The conditional and joint characteristic functions

In this section we will investigate the two point statistics of the filtering process. First we will consider the probability of finding a value E at time t given that the value was E_0 at an earlier time which we take (without loss of generality) as $t=0$. Some care needs to be taken here since (as noted by Pawula [10]) specifying a value of E_0 does not give a complete description of the system at $t=0$. The subsequent time evolution depends on whether the telegraph signal is in the +1 or -1 state at $t=0$. We will start by treating the two cases separately and use E_0^+ to indicate that the signal is in the +1 state and E_0^- for the -1 state.

By differentiating and combining (8) and (9), a single second-order equation can be found for the combined characteristic function $C = C^+ + C^-$:

$$k^2\frac{\partial^2 C}{\partial k^2} + 2\eta k\frac{\partial C}{\partial k} + k^2 C = (1-2\eta)\frac{\partial C}{\partial t} - \frac{\partial^2 C}{\partial t^2} - 2k\frac{\partial^2 C}{\partial k\partial t}. \tag{10}$$

The general solution of this equation is straightforward; making the changes of variables to $z = \exp(t)$ and $y = k\exp(-t)$ reduces it to the standard form:

$$z^2\frac{\partial^2 C}{\partial z^2} + 2\eta z\frac{\partial C}{\partial z} + y^2 z^2 C = 0 \tag{11}$$

which has solutions (in terms of the original variables)

$$C(k,t) = \varepsilon^{\eta-\frac{1}{2}}(A(k\varepsilon)J_{\eta-\frac{1}{2}}(k) + B(k\varepsilon)Y_{\eta-\frac{1}{2}}(k)) \tag{12}$$

where $\varepsilon = \exp(-t)$, J_v is the Bessel function of the first kind of order v, and Y_v is the Bessel function of the second kind. A and B are arbitrary functions, with argument $k\varepsilon$.

Boundary conditions are provided by the value of C and its first derivative with respect to time at $t = 0$. E takes the value E_0 at $t = 0$, which gives $C(k, t = 0) = \exp(-\mathrm{i}kE_0)$. The value of the time derivative at zero can be found from equations (8), (9), and depends on the state of the telegraph signal:

$$\begin{aligned} \frac{\partial C}{\partial t}\Big|_{E_0^+} &= \mathrm{i}k(E_0 - 1)\mathrm{e}^{-\mathrm{i}kE_0} \\ \frac{\partial C}{\partial t}\Big|_{E_0^-} &= \mathrm{i}k(E_0 + 1)\mathrm{e}^{-\mathrm{i}kE_0}. \end{aligned} \tag{13}$$

Using these boundary conditions it is not difficult to show that

$$\begin{aligned} C(k, t|E_0^+) = \frac{\pi k\varepsilon^{\eta+\frac{1}{2}}\mathrm{e}^{-\mathrm{i}kE_0\varepsilon}}{2}&(Y_{\eta-\frac{1}{2}}(k)[J_{\eta+\frac{1}{2}}(k\varepsilon) - \mathrm{i}J_{\eta-\frac{1}{2}}(k\varepsilon)] \\ &-J_{\eta-\frac{1}{2}}(k)[Y_{\eta+\frac{1}{2}}(k\varepsilon) - \mathrm{i}Y_{\eta-\frac{1}{2}}(k\varepsilon)]). \end{aligned} \tag{14}$$

The corresponding expression for the case in which the telegraph signal is in the -1 state at $t = 0$ is

$$\begin{aligned} C(k, t|E_0^-) = \frac{\pi k\varepsilon^{\eta+\frac{1}{2}}\mathrm{e}^{-\mathrm{i}kE_0\varepsilon}}{2}&(Y_{\eta-\frac{1}{2}}(k)[J_{\eta+\frac{1}{2}}(k\varepsilon) + \mathrm{i}J_{\eta-\frac{1}{2}}(k\varepsilon)] \\ &-J_{\eta-\frac{1}{2}}(k)[Y_{\eta+\frac{1}{2}}(k\varepsilon) + \mathrm{i}Y_{\eta-\frac{1}{2}}(k\varepsilon)]). \end{aligned} \tag{15}$$

For large t (14) and (15) become equal and independent of t and E_0. This is because the filter has a limited memory. The time-independent first-order probability density can be found from (14) or (15) by taking $t \to \infty$, which gives

$$C(k) = \Gamma\left(\frac{1}{2} + \eta\right)\left(\frac{k}{2}\right)^{\frac{1}{2}-\eta} J_{\eta-\frac{1}{2}}(k). \tag{16}$$

Inverse transformation yields the well known result

$$P(E)\begin{cases} = \dfrac{\Gamma(\frac{1}{2}+\eta)}{\sqrt{\pi}\,\Gamma(\eta)}(1 - E^2)^{\eta-1} & \text{for} \quad |E| < 1 \\ = 0 & \text{otherwise} \end{cases} \tag{17}$$

where Γ is the Euler gamma function. The densities P^+ and P^- are most easily found by using the relationship $(1 - E)P^+(E) = (1 + E)P^-(E)$. This follows from the fact that the filter output passes through a given value of E the same number of times in the negative direction as it does in the positive direction, thus the probabilities $P^+(E)$ and $P^-(E)$ only differ due to the different rates of change, which are (from (2)) $(1 - E)$ and $(1 + E)$ respectively. Substitution into (5) and (6) yields $P^+(E) = (1 + E)P(E)/2$ and $P^-(E) = (1 - E)P(E)/2$. From these results one can see that the following conditional probabilities apply:

$$\begin{aligned} P(T = +1|E) &= (1 + E)/2 \\ P(T = -1|E) &= (1 - E)/2. \end{aligned} \tag{18}$$

These can be used to combine (14) and (15) into a single *conditional* characteristic function which is independent of the state of the telegraph signal at $t = 0$:

$$\begin{aligned} C(k, t|E_0) = \frac{\pi k\varepsilon^{\eta+\frac{1}{2}}\exp(-\mathrm{i}k\varepsilon E_0)}{2}&(Y_{\eta-\frac{1}{2}}(k)J_{\eta+\frac{1}{2}}(k\varepsilon) - J_{\eta-\frac{1}{2}}(k)Y_{\eta+\frac{1}{2}}(k\varepsilon) \\ &+\mathrm{i}E_0(J_{\eta-\frac{1}{2}}(k)Y_{\eta-\frac{1}{2}}(k\varepsilon) - Y_{\eta-\frac{1}{2}}(k)J_{\eta-\frac{1}{2}}(k\varepsilon))). \end{aligned} \tag{19}$$

From this we can calculate the *joint* characteristic function. This is the Fourier transform of the joint probability density, which is the probability of obtaining E_0 at $t = 0$ and E at

t. We multiply equation (19) by $P(E_0)$ and take the Fourier transform, with new transform variable k', which gives

$$\begin{aligned}C(k',k) = {} & \frac{\pi\Gamma(\frac{1}{2}+\eta)k(\frac{k'+k\varepsilon}{2})^{\frac{1}{2}-\eta}\varepsilon^{\eta+\frac{1}{2}}}{2} \\ & \times\{J_{\eta-\frac{1}{2}}(k'+k\varepsilon)[Y_{\eta-\frac{1}{2}}(k)J_{\eta+\frac{1}{2}}(k\varepsilon) - J_{\eta-\frac{1}{2}}(k)Y_{\eta+\frac{1}{2}}(k\varepsilon)] \\ & +J_{\eta+\frac{1}{2}}(k'+k\varepsilon)[J_{\eta-\frac{1}{2}}(k)Y_{\eta-\frac{1}{2}}(k\varepsilon) - Y_{\eta-\frac{1}{2}}(k)J_{\eta-\frac{1}{2}}(k\varepsilon)]\}. \end{aligned} \tag{20}$$

4. Distributions and correlation coefficients

Closed forms for the characteristic functions corresponding to the conditional and joint densities given in the last section have not previously been published. However, some expressions for the densities themselves have already appeared in the literature. In this section we shall indicate how the probability densities characterizing E of equation (1) can be obtained from the results of the last section, referring as appropriate to previous work.

The single interval equilibrium density (17) appears to have been first derived in the present context by Wonham and Fuller [7] using the method of moments. The present authors also used this method [6] which exploits the factorization properties of the N-fold correlation function of a random telegraph signal [7, 28] and symmetries within the multiple integral which has to be evaluated. The result obtained in this way is

$$\begin{aligned}\langle E^{2N}\rangle &= \frac{(2N)!\Gamma\left(\frac{1}{2}(1+\eta)\right)}{2^{2N}N!\Gamma\left(N+\frac{1}{2}(1+\eta)\right)} \\ \langle E^{2N+1}\rangle &= 0\end{aligned} \tag{21}$$

and it is not difficult to demonstrate directly that these are the moments of the symmetric single interval distribution (17). Conversely, having calculated the moments (21) from first principles one can construct the characteristic function

$$\langle \exp(\mathrm{i}kE)\rangle = \sum_{N=0}^{\infty}\frac{(-1)^N}{(2N)!}\langle E^{2N}\rangle \tag{22}$$

leading to result (16).

In order to evaluate the conditional and joint probability densities from the results of the last section we first note that equation (19) may be expressed in the form

$$C(k,t|E_0) = \frac{1}{2}\pi\varepsilon^{\eta+\frac{1}{2}}\exp(-\mathrm{i}k\varepsilon E_0)\left(\frac{\eta-\frac{1}{2}}{\varepsilon} - \frac{\partial}{\partial\varepsilon} - \mathrm{i}kE_0\right)p_\eta(k,\varepsilon)$$

where

$$p_\eta(k,\varepsilon) = Y_{\eta-\frac{1}{2}}(k)J_{\eta-\frac{1}{2}}(k\varepsilon) - J_{\eta-\frac{1}{2}}(k)Y_{\eta-\frac{1}{2}}(k\varepsilon). \tag{23}$$

The simpler characteristic function conditional on $E_0 = 0$ at time $t = 0$ was apparently first derived by McFadden ([9]—given in a slightly different form) and again more recently by Morita [29], but the corresponding distribution was derived later from the real-space partial differential equations by Wonham [8] and therefore only an *indirect* proof of their equivalence exists. A careful search of the literature has failed to reveal the Fourier transform of the relatively simple combination of Bessel functions appearing in expression (23). However, defining

$$f_v(E) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}k\,\exp(\mathrm{i}kE)[Y_v(k)J_v(k\varepsilon) - J_v(k)Y_v(k\varepsilon)] \tag{24}$$

we show in appendix D that a result may be obtained in terms of Legendre functions as follows:

$$f_v(E) \begin{cases} = \dfrac{1}{\pi\sqrt{\varepsilon}} P_{v-\frac{1}{2}}\left(\dfrac{1+\varepsilon^2-E^2}{2\varepsilon}\right) & |E| \leqslant 1-\varepsilon \\ = 0 & \text{otherwise.} \end{cases} \tag{25}$$

With a little manipulation the Fourier transform of (23) can now be expressed, using this result, in the form

$$P(E,t|E_0) \begin{cases} = \dfrac{\eta\varepsilon^{\eta-1}}{2(1-x^2)}[(X-x)P_\eta(x) + (1-xX)P_{\eta-1}(x)] + \varepsilon^\eta \Delta \\ \qquad \text{for} \quad |E_0| \leqslant 1, |E-\varepsilon E_0| \leqslant 1-\varepsilon \\ = 0 \\ \qquad \text{otherwise} \end{cases} \tag{26}$$

where

$$X = \varepsilon - EE_0 + \varepsilon E_0^2 \qquad x = [1+\varepsilon^2 - (E-\varepsilon E_0)^2]/2\varepsilon$$
$$\Delta = \tfrac{1}{2}[(1+E_0)\delta(E-\varepsilon E_0 - 1+\varepsilon) + (1-E_0)\delta(E-\varepsilon E_0 + 1 - \varepsilon)].$$

The earlier result of Pawula [10] is expressed in term of hypergeometric functions but using linear transformation formulae and Gauss's relations for contiguous functions can be shown (see appendix A) to be identical to formula (26). The joint probability density is obtained immediately by multiplying this expression by the single interval equilibrium density (17) for E_0.

A number of useful results for the low-order moments and correlation functions can be derived from the above formulae or directly from the definition (1) using the factorization properties of the telegraph signal correlations. For example, the fourth normalized moment of E can be written down immediately from equation (17):

$$\frac{\langle E^4\rangle}{\langle E^2\rangle^2} = 3\frac{\eta+1}{\eta+3}. \tag{27}$$

This provides a simple measure of the integrating effect of the filter, tending to unity in the broadband limit, $\eta \to 0$, when the filter has no effect so that $E^2 \propto T^2 = 1$ and to the one-dimensional Gaussian value of 3 in the narrowband limit.

The two-time correlation functions for the process can be calculated from (20) or (26), or directly from the definition (1) using the factorization properties of the telegraph signal correlations. Thus we obtain:

$$g^{(1)}(t) = \frac{\langle E(0)E(t)\rangle}{\langle E^2\rangle} = \frac{\exp(-\eta|t|) - \eta\exp(-|t|)}{1-\eta} \tag{28}$$

$$g^{(2)}(t) = \frac{\langle E^2(0)E^2(t)\rangle}{\langle E^2\rangle^2} = 1 + \frac{2\eta\exp(-|t|)[2\exp(-\eta|t|) - (1+\eta)\exp(-|t|)]}{(3+\eta)(1-\eta)}. \tag{29}$$

It is not difficult to verify that formula (29) reduces to (27) in the limit $t \to 0$.

5. The detuned case

If the telegraph signal is detuned from the centre of the filter ($\omega \neq 0$), E, as defined by equation (1) becomes complex; thus the changes in E can be considered as a two-dimensional random walk in the complex plane. Letting $E = x + \mathrm{i}y$, we can describe the operation of the filter by a pair of differential equations:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -x + \xi y \pm 1 \tag{30}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -y - \xi x \tag{31}$$

where $\xi = \frac{\omega}{\lambda}$. The probability densities are functions of both x and y, e.g. $p^{+}(x, y, t)\,\mathrm{d}x\,\mathrm{d}y$. Differential equations for these densities can be derived in the same way as in the one dimensional case. In this case it is the change in differential area $\mathrm{d}x\,\mathrm{d}y$ with time which needs to be taken into account. Following the same procedure of retaining terms to first order in δt leads to the following pair of equations:

$$(2-\eta)p^{+} + \eta p^{-} + (x - \xi y - 1)\frac{\partial p^{+}}{\partial x} + (y + \xi x)\frac{\partial p^{+}}{\partial y} = \frac{\partial p^{+}}{\partial t} \tag{32}$$

$$(2-\eta)p^{-} + \eta p^{+} + (x - \xi y + 1)\frac{\partial p^{-}}{\partial x} + (y + \xi x)\frac{\partial p^{-}}{\partial y} = \frac{\partial p^{-}}{\partial t}. \tag{33}$$

In the case of $\omega = 0$ these reduce to equations (5) and (6) as follows: the two-dimensional probability density $p(x, y, t)\,\mathrm{d}x\,\mathrm{d}y$ is reduced to the one-dimensional density $p(x, t)\,\mathrm{d}x$ by integration over the y coordinate. For example, equation (32) is rewritten as

$$(1-\eta)p^{+} + \eta p^{-} + (x - 1)\frac{\partial p^{+}}{\partial x} + \frac{\partial y p^{+}}{\partial y} = \frac{\partial p^{+}}{\partial t}. \tag{34}$$

The substitution $p(x, y) = p(x)\delta(y)$ (where δ is the Dirac delta function) is made and, when the integration over y is carried out, the final term on the left-hand side of (34) gives zero, resulting in equation (5).

These are significantly more complicated than the $\omega = 0$ equations, and we have not succeeded in solving for the time-independent density, even for special values of η. The chances of finding closed form solutions for the conditional and joint distributions or their characteristic functions appear to be slim.

It is possible to calculate the low-order moments and correlation functions directly from equation (1), and for the second and fourth moments one obtains

$$\langle |E|^2 \rangle = \frac{1+\eta}{(1+\eta)^2 + \xi^2} \tag{35}$$

and

$$\frac{\langle |E|^4 \rangle}{\langle |E|^2 \rangle^2} = \frac{2(3+\eta)[(1+\eta)^2+\xi^2]}{(1+\eta)[(3+\eta)^2+\xi^2]} + \frac{[(1+\eta)^2+\xi^2][(1+\eta)(3+\eta) - \xi^2(5+2\eta)]}{(1+\xi^2)(1+\eta)^2[(3+\eta)^2+\xi^2]} \tag{36}$$

whilst the first-order coherence function may be expressed in the form

$$\langle E(0)E^{*}(t) \rangle = \frac{1}{1-\eta-\mathrm{i}\xi}\left[\frac{\exp(-\eta|t|)}{1+\eta+\mathrm{i}\xi} - \frac{\eta\exp(-|t| + \mathrm{i}\xi t)}{1+\eta-\mathrm{i}\xi}\right]. \tag{37}$$

The second-order coherence function can also be calculated from equation (1) through a lengthy and tedious calculation. The rather complicated result is analytically opaque and is therefore presented only in an appendix (appendix B). However, some numerical results are discussed in the next section.

A further quantity of interest is the degree of correlation between the outputs of two filters centred at different frequencies. As we have discussed elsewhere [6], this arrangement is analogous to a light scattering configuration in which the correlations in light relating to different scattering vectors are measured. These are known to exhibit spatial 'memory' effects in which high speckle correlation is observed for certain combinations of incident and

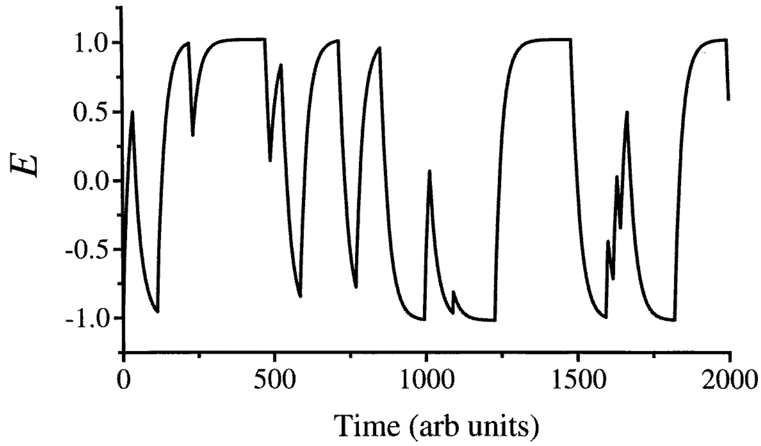

Figure 2. An example of the time evolution of E. Here the time is in arbitrary units, and the dimensionless transition rate is $\eta = 1$.

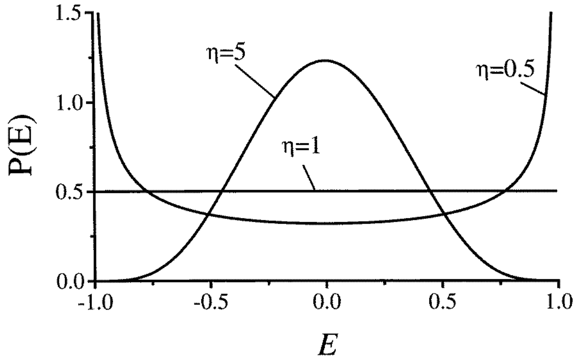

Figure 3. Plots of the single-interval probability density, given by equation (17), for different values of the dimensionless transition rate η.

scattered directions [26]. Unfortunately, calculation of the relevant correlation function from equation (1) again results in a lengthy and opaque analytical formula. Numerical results will be presented in the next section but the details of the calculation and formula will be reserved for appendix C.

6. Discussion

Some numerical simulations of the Lorentzian filtered telegraph signal are shown in figure 2 (see [6]). Examples of the single interval equilibrium density (17), which is plotted in figure 3, are commonly encountered in the literature. The distribution progresses from a pair of delta functions at ± 1 when η is zero, through a uniform distribution when $\eta = 1$, to a Gaussian shape at large values of the parameter. According to equation (27) its normalized second moment increases monotonically from one to three over the same range. The case $\eta = \frac{1}{2}$ is the well known 'U'-shaped distribution of the amplitude of a randomly phased sine wave. A number of stochastic models lead to this kind of distribution, for example, the *Jacobi* process [30]. However, the higher-order joint statistics and correlation properties for this Markov process are different from those of the Lorentzian-filtered telegraph signal studied here.

The relationship to the random phasor problem suggests an investigation of the more general d-dimensional case. We assume that

$$\vec{u} = (a_1, a_2, a_3, \ldots, a_d)$$

is a randomly oriented vector in a d-dimensional space with Cartesian components a_{i}. The characteristic function,

$$C(\vec{u}) = \langle \exp(-\mathrm{i}\vec{u} \cdot \vec{u}) \rangle$$
$$P(a_1, a_2, \ldots, a_d) = \frac{1}{(2\pi)^d} \int \mathrm{d}\vec{u}\; C(\vec{u}) \exp\left(\mathrm{i} \sum_{n=1}^{d} u_n a_n\right) \tag{38}$$

can be calculated by writing $\vec{u} \cdot \vec{u} = ua \cos\theta$ where the components of the phasor are

$$\begin{aligned}
a_1 &= a \cos\theta \\
a_2 &= a \sin\theta \cos\phi_1 \\
a_3 &= a \sin\theta \sin\phi_1 \cos\phi_2 \\
&\vdots \\
a_d &= a \sin\theta \sin\phi_1 \sin\phi_2 \ldots \sin\phi_{d-2} \cos\phi_{d-1}.
\end{aligned}$$

Thus

$$\begin{aligned}
C(\vec{u}) &= \int_0^{\pi} \mathrm{d}\theta \sin^{d-2}\theta \exp(-\mathrm{i}ua \cos\theta) \\
&= \sqrt{\pi}\,\Gamma\left(\frac{d-2}{2}\right) \frac{J_{\frac{d}{2}-1}(ua)}{\left(\frac{1}{2}ua\right)^{\frac{d}{2}-1}}
\end{aligned} \tag{39}$$

which is similar to equation (16). The probability density of any component of the phasor can be calculated from this result by substituting into equation (38) and setting all the $\{u_n\}$ equal to zero save one. This gives

$$P(a) = \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)} (1 - a^2)^{\frac{d-3}{2}} \tag{40}$$

which is identical with the distribution (17) if we make the identification

$$\eta = (d-1)/2. \tag{41}$$

Clearly only the integer and half-integer values of η correspond to an integer number of dimensions in the phasor problem and it is interesting that the present model appears to interpolate result (40) to the case where d may be fractional.

The transformation of variables

$$f = \frac{E}{\sqrt{1 - E^2}} \tag{42}$$

applied to the single-interval equilibrium density (17) obtains the class of distributions

$$P(f) = \frac{\Gamma(\eta + \frac{1}{2})}{\sqrt{\pi}\,\Gamma(\eta)} \frac{1}{(1 + f^2)^{\eta + \frac{1}{2}}} \qquad \text{for} \quad |f| < \infty. \tag{43}$$

Here f would be the cotangent of the angle between a random phasor moving in a d-dimensional space and a given direction for example, if (41) was satisfied. These student's t-distributions with 2η degrees of freedom have attracted new interest recently because of their power-law behaviour at large arguments and have been derived by minimizing a constrained

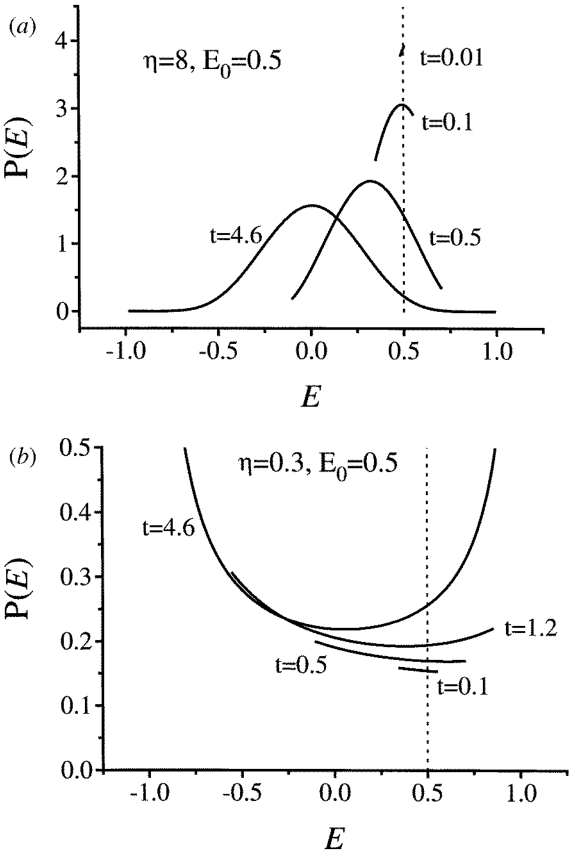

Figure 4. The probability density function (omitting the delta function parts) at time t for various values of t conditional on E being equal to 0.5 at $t = 0$. (a) For a dimensionless transition rate η equal to 8. (b) For $\eta = 0.5$.

generalized entropy function [31]. The special case $\eta = \frac{1}{2}$ is the Cauchy or Laplace distribution, which is a member of the Levy-stable class and has an exponential characteristic function [27]. In the present context, where f is interpreted as a scalar variable, other members of the class are not stable as their characteristic functions are modified Bessel functions with the same form as K-distributions [32]. However, if f were the amplitude of a vector in a D-dimensional space, then a distribution of the type (43) (though with a different normalization) would be stable if $\eta = \frac{D}{2}$.

The time evolution of the statistics governed by formula (26) is illustrated in figures 4(a) and (b) for two values of η. The delta function parts are omitted and the density is plotted over the region for which it is non-zero. Both plots start at $E_0 = 0.5$, and show how the distribution becomes symmetric about $E = 0$ for long times. The functional form is highly dependent on the parameter η even for short delay times t (note, however, that the delta function bits, which dominate the densities for small t, are omitted).

We note that when $\eta = \frac{1}{2}$, the result of (20) may be expressed in terms of elliptic integrals. As pointed out by Pawula [10], the conditional distribution reduces to a particularly simple

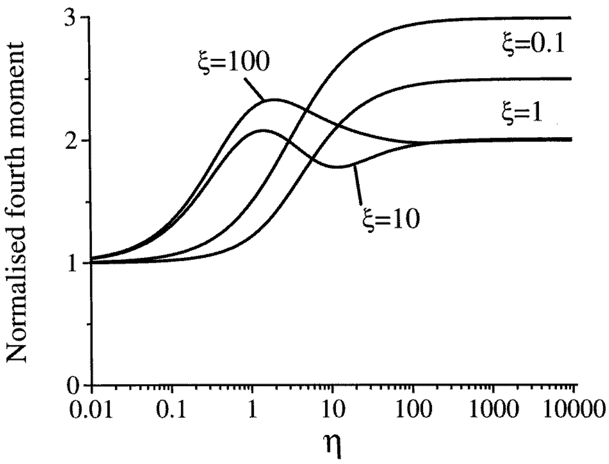

Figure 5. The normalized fourth moment of $|E|$ (equation (36)) as a function of the dimensionless transition rate η, for different values of the dimensionless detuning parameter ξ.

form when $\eta = 1$:

$$P(E, t|E_0) \begin{cases} = \dfrac{1}{2} + \varepsilon\Delta & \text{for} \quad |E_0| \leqslant 1, |E - \varepsilon E_0| \leqslant 1 - \varepsilon \\ = 0 & \text{otherwise} \end{cases} \tag{44}$$

where $\varepsilon = \exp(-|t|)$ as before and Δ is defined in equation (26). According to equation (17) this need only be multiplied by a further factor of $\frac{1}{2}$ to obtain the joint distribution. (44) is characterized by a single time constant because the filter response time is matched to the telegraph signal switching time. This is manifest in the first- and second-order coherence functions (28), (29) which reduce for this case to

$$\begin{aligned} g^{(1)}(t) &= (1 + |t|)\exp(-|t|) \\ g^{(2)}(t) &= 1 + \tfrac{1}{2}(1 + 2|t|)\exp(-2|t|). \end{aligned} \tag{45}$$

Note that the leading-order terms in an expansion of the first-order correlation function about $t = 0$ are of the form $1 - t^2 + |t|^3 + \cdots$, indicating the sub-fractal nature of E. In fact, this behaviour is found for all values of η and relates to the fact that E is only once differentiable, the telegraph signal itself behaving like a Brownian fractal with an outer but no inner scale.

It is evident from formula (26) that the conditional density reduces to a simple combination of Legendre polynomials when η is an integer and it is then possible to express the result algebraically. Thus, for the case $\eta = 2$ we obtain

$$P(E, t|E_0) \begin{cases} = \tfrac{3}{2}(1 - E^2) + 4\varepsilon E E_0 + \tfrac{1}{2}(1 - 5E_0^2) + \varepsilon^2\Delta \\ \qquad \text{for} \quad |E_0| \leqslant 1, |E - \varepsilon E_0| \leqslant 1 - \varepsilon \\ = 0 \\ \qquad \text{otherwise.} \end{cases} \tag{46}$$

The coherence function (28) is clearly characterized by two different exponential decays in this case.

In the detuned situation, the fourth-order normalized moment of E approaches the value $2 + 1/(1 + \xi^2)$ in the limit when $\gamma \to \infty$, i.e. as the telegraph signal crossing rate increases with all other frequencies fixed. When the detuning is small this expression reduces to 3—the

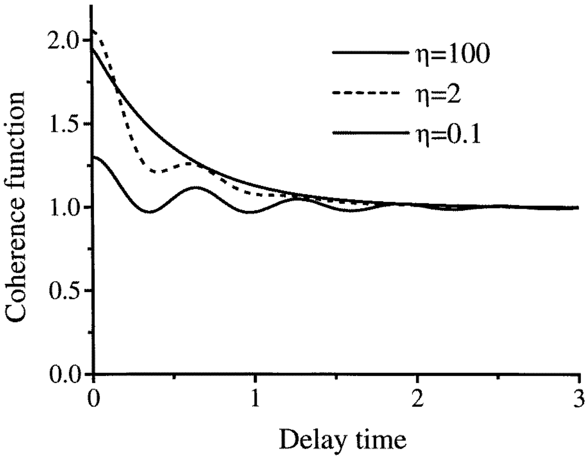

Figure 6. The coherence function (equation (C.3), normalized by equation (35)) as a function of time for detuning parameter $\xi = 10$, and different values of the dimensionless transition rate η.

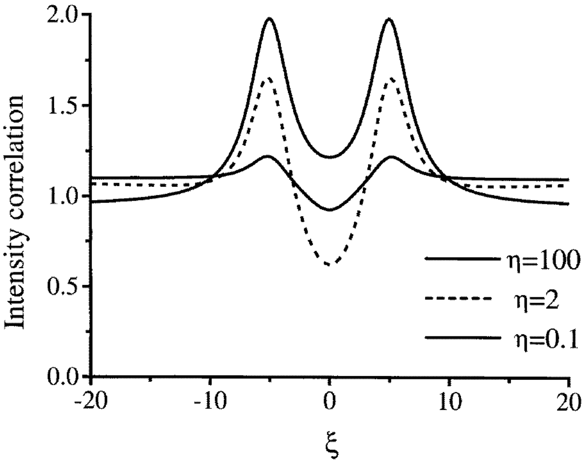

Figure 7. Intensity correlation function for two different frequencies, plotted as a function of ξ for $\xi = 5$, and three different values of the dimensionless transition rate η.

result expected for a linear one-dimensional Gaussian process, whilst when the detuning is large it reduces to 2 corresponding to a circular complex Gaussian process. In between these limits the process is elliptical. Some care is required in the analysis of limiting cases. For example, in a previous paper [6] we have shown by scaling parameters with γ rather than λ that if ω is non-zero the narrow band limit, $\lambda \to 0$, *always* leads to a normalized fourth moment of 2. Figure 5 shows that there can be a weak maximum in the degree of fluctuation as a function of crossing rate rather than the monotonic behaviour found when E is real. This is also manifest in the height of the peak at zero delay of the coherence functions plotted in figure 6 for $\xi = 10$, which is slightly in excess of 2 for $\eta = 2$. In this detuned case oscillations are observed when the telegraph signal crossing rate is on the order of, or smaller than, the offset frequency. However, these disappear as the crossing rate increases.

Figure 7 shows the correlation between intensities obtained using filters centred at different frequencies. Peaks are observed when the filters are centred at the same frequency or at frequencies symmetrically placed about zero. This phenomenon is related to the simplest of the so-called 'memory' effects which has figured prominently in recent years in the literature on the spatial structure of speckle patterns in multiple scattering configurations [26]. In the present work it is a trivial consequence of the fact that, from equation (1), the square modulus of E is independent of the sign of the frequency shift, ξ (see equation (C.1)). Note that related effects were also predicted and observed in earlier work on the detection of hidden gratings [33].

In the Gaussian (narrowband) limit the peak of the correlation is 2 and saturation occurs at a value of unity as expected when the frequency separation becomes large. When η is small, however, there is some residual correlation at large frequency separations. This is because the filter integrates over only a few flips of the telegraph signal in this limit and these generate variations in the signal spanning a wide frequency range. Similarly, the anti-correlation predicted when the crossing rate and filter bandwidth are comparable (figure 7) is caused by flips of the telegraph signal directing energy into frequency bands which vary as the memory function sweeps over the input signal.

Finally, we note that a number of other results on the distribution of slope, level crossings, maxima and minima and specular points have been given by Pawula [11].

7. Summary

In this paper we have presented a simple formulation of the equations characterizing the statistics of a Lorentzian filtered random telegraph signal. These we have solved for the characteristic function and conditional probability density of the fluctuations. We have discussed the results in the context of previous work. We have illustrated both individual realizations of the process and its statistical properties by a combination of numerical and analytical techniques and drawn attention to properties which may be of particular current interest. Whilst the original motivation for our investigation lies in the area of light scattering and laser physics, the model we have studied and other related dichotomous noise driven systems which can be studied using the same approach, have potential as stochastic models in a wide range of applications as witnessed by the large recent literature on the subject.

Acknowledgment

We are indebted to M V Berry for suggesting the method of approach used in appendix D.

Appendix A

We show here how Pawula's original solution for the conditional distribution of a Lorentzian filtered telegraph wave in the tuned case may be expressed in terms of Legendre functions. The transformation formulae for hypergeometric functions used in the derivation and their relationship to the Legendre functions are given in [34], chapters 8 and 15.

According to Pawula [10],

$$P(E,t|E_0) = \frac{\eta}{4}\left(\frac{1-w}{\varepsilon}\right)^{1-\eta}[(1+\varepsilon-EE_0+\varepsilon E_0^2)F_1(w)+\eta(1-\varepsilon+EE_0-\varepsilon E_0^2)F_2(w)] \\ +\varepsilon^\eta\Delta \qquad \text{for} \quad |E_0|\leqslant 1, |E-\varepsilon E_0|\leqslant 1-\varepsilon \tag{A.1}$$

where

$$
\begin{aligned}
w &= \frac{(1-\varepsilon)^2 - (E-\varepsilon E_0)^2}{(1+\varepsilon)^2 - (E-\varepsilon E_0)^2} \\
F_1(w) &= {}_2F_1(1-\eta, -\eta; 1; w) = (1-w)^{\eta-1}\, {}_2F_1(1-\eta, 1+\eta; 1; w/(w-1)) \\
F_2(w) &= {}_2F_1(1-\eta, 1-\eta; 2; w) = (1-w)^{\eta-1}\, {}_2F_1(1-\eta, 1+\eta; 2; w/(w-1)).
\end{aligned}
\tag{A.2}
$$

Now, from [34] we have

$$
\begin{aligned}
{}_2F_1(1-\eta, 1+\eta; 2; z) &= \frac{-(1-z)^{1+\eta}}{\eta^2} \frac{\mathrm{d}}{\mathrm{d}z} (1-z)^{-\eta}\, {}_2F_1(-\eta, 1+\eta; 1; z) \\
&= \frac{-(1-z)^{1+\eta}}{\eta^2} \frac{\mathrm{d}}{\mathrm{d}z} (1-z)^{-\eta} P_\eta(1-2z) \\
&= -\frac{1}{2\eta z}[P_\eta(1-2z) - P_{\eta-1}(1-2z)]
\end{aligned}
\tag{A.3}
$$

and

$$
\begin{aligned}
{}_2F_1(1-\eta, 1+\eta; 1; z) &= \frac{\mathrm{d}}{\mathrm{d}z} z\, {}_2F_1(1-\eta, 1+\eta; 2; z) \\
&= -\frac{1}{2\eta} \frac{\mathrm{d}}{\mathrm{d}z}[P_\eta(1-2z) - P_{\eta-1}(1-2z)] \\
&= \frac{1}{2(1-z)}[P_\eta(1-2z) + P_{\eta-1}(1-2z)].
\end{aligned}
\tag{A.4}
$$

Substitution of (A.3) and (A.4) into (A.1) and (A.2) yields our equation (26).

Appendix B

In this appendix we outline the calculation of the second-order coherence function for the detuned case. We take advantage of the factorization properties of the random telegraph signal [28], in particular the relation

$$
\langle T(x_1)T(x_2)T(x_3)T(x_4)\rangle = \exp[-\eta(x_4 - x_3 + x_2 - x_1)] \qquad \text{for} \quad x_1 < x_2 < x_3 < x_4. \tag{B.1}
$$

The required statistic may be written as

$$
\begin{aligned}
\langle |E(0)|^2 |E(t)|^2 \rangle = \int_{-\infty}^{0} \mathrm{d}t_1 \int_{-\infty}^{0} \mathrm{d}s_1 \int_{-\infty}^{t} \mathrm{d}t_2 \int_{-\infty}^{t} \mathrm{d}s_2 \cos\xi(t_1 - s_1)\cos\xi(t_2 - s_2) \\
\times \exp(t_1 + s_1 + t_2 + s_2 - 2t)\langle T(t_1)T(t_2)T(s_1)T(s_2)\rangle.
\end{aligned}
\tag{B.2}
$$

Exploiting the symmetry properties of the integrals, they can be expressed in the form

$$
\begin{aligned}
\langle |E|^4\rangle \exp(-2t) + 4\int_0^t \mathrm{d}t_2 \bigg\{ \int_{-\infty}^{0} \mathrm{d}s_2 \int_{-\infty}^{s_2} \mathrm{d}t_1 \int_{-\infty}^{t_1} \mathrm{d}s_1 + \int_{-\infty}^{0} \mathrm{d}s_2 \int_{s_2}^{0} \mathrm{d}t_1 \int_{-\infty}^{s_2} \mathrm{d}s_1 \\
+ \int_{-\infty}^{0} \mathrm{d}s_2 \int_{s_2}^{0} \mathrm{d}t_1 \int_{s_2}^{t_1} \mathrm{d}s_1 + \int_{0}^{t_2} \mathrm{d}s_2 \int_{-\infty}^{0} \mathrm{d}t_1 \int_{-\infty}^{t_1} \mathrm{d}s_1 \bigg\} I(t_1 \ldots s_2)
\end{aligned}
$$

where I is the integrand appearing in (B.2). The variables of integration in each of these four integrals is now ordered so that the factorization theorem (B.1) can be applied to the integrand. The calculation of these contributions is then straightforward though tedious and will not be repeated here. The final result is

$$
\langle |E(0)|^2 |E(t)|^2\rangle = \langle |E|^4\rangle \exp(-2t) + 4(I_1 + I_2 + I_3 + I_4)
$$

with

$$
\begin{aligned}
I_1 &= \tfrac{1}{2}abd(1+\eta)\{(C[(3+\eta)(1-\eta)+\xi^2]+2(1+\eta)S)\}\\
I_2 &= ad\left\{C\left[\frac{2(1-\eta)+(2-\eta)\xi^2}{4(1+\xi^2)}-\frac{(1-\eta)(3+\eta)+\xi^2}{b^{-1}}\right]\right.\\
&\qquad \left.+S\left[\frac{1+\xi^2+\eta}{4(1+\xi^2)}-\frac{2(1+\eta)}{b^{-1}}\right]\right\}\\
I_3 &= Cad^2\times\left\{\frac{\xi^4+(1-\eta)(7+5\eta)\xi^2+2(3+\eta)(1-\eta^2)}{b^{-1}}\right.\\
&\qquad \left.-\frac{(2-3\eta)\xi^4+(1-\eta)[(4-\eta-\eta^2)\xi^2+2(1-\eta^2)]}{4(1+\xi^2)}\right\}\\
&+Sad^2\left\{\frac{(1+3\eta)\xi^2+(1-\eta)(1+10\eta+5\eta^2)}{b^{-1}}-\frac{\xi^4+(1-\eta)[(2+3\eta)\xi^2+(1+\eta)^2]}{4(1+\xi^2)}\right\}\\
I_4 &= \frac{a^2(1+\eta)^2(1-\mathrm{e}^{-2t})}{4}-\frac{da^2(1+\eta)}{2}[C(1-\eta^2+\xi^2)+2\eta S]\\
a &= 1/[(1+\eta)^2+\xi^2] \qquad b = 1/[(3+\eta)^2+\xi^2] \qquad d = 1/[(1-\eta)^2+\xi^2]\\
C &= \exp[-(1+\eta)t]\cos\xi t-\exp(-2t) \qquad S = \xi\exp[-(1+\eta)t]\sin\xi t.
\end{aligned}
\tag{B.3}
$$

Appendix C

In this appendix we calculate the correlation between the output intensities from two filters centred at different frequencies, assuming that the input to each filter is the same realization of the telegraph signal. The starting point is the definition (1) of the text and we shall exploit the factorization property (B.1). According to (1) the output intensity from a filter centred at normalized frequency ξ is given by

$$
|E_\xi^2| = \int_{-\infty}^{t}\mathrm{d}t_1\int_{-\infty}^{t}\mathrm{d}t_2 T(t_1)T(t_2)\exp[t_1+t_2+\mathrm{i}\xi(t_1-t_2)]. \tag{C.1}
$$

Here the filter time constant has been set equal to unity for simplicity. Applying stationarity and symmetry arguments the required correlation function may be written as

$$
\langle|E_\xi|^2|E'_\xi|^2\rangle = \int_0^\infty \mathrm{d}t_1\ldots\mathrm{d}t_4\langle T(t_1)\ldots T(t_4)\rangle\exp(t_1+\cdots t_4)\cos\xi(t_1-t_2)\cos\xi'(t_3-t_4). \tag{C.2}
$$

This multiple integral may be expressed as the sum of 24 terms in which the integration variables are differently ordered. However, since the kernel is invariant under exchange of t_1 with t_2 and t_3 with t_4 this is reduced by a factor of four. Further symmetries eventually lead to

$$
\langle|E_\xi|^2|E'_\xi|^2\rangle = 4\sum_{n=1}^{3}[I_n(\xi,\xi')+I_n(\xi',\xi)] \tag{C.3}
$$

where the I_n are integrals of the kernel in equation (C.2) over the regions $t_1<t_2<t_3<t_4$, $t_1<t_3<t_2<t_4$, $t_1<t_3<t_4<t_2$. These may be evaluated with the help of (B.1) and we obtain

$$
\begin{aligned}
I_1(\xi,\xi') &= \tfrac{1}{8}a(\xi)b(\xi')(1+\eta)(3+\eta)\\
I_2(\xi,\xi') &= \tfrac{1}{8}a(\xi')b(\xi)[f(\xi,\xi')c(\xi,\xi')+f(\xi,-\xi')c(\xi,-\xi')]\\
I_3(\xi,\xi') &= \tfrac{1}{4}a(\xi)b(\xi)[g(\xi,\xi')c(\xi,\xi')+g(\xi,-\xi')c(\xi,-\xi')]
\end{aligned}
\tag{C.4}
$$

where

$$a(\xi) = [\xi^2 + (1+\eta)^2]^{-1} \qquad b(\xi) = [\xi^2 + (3+\eta)^2]^{-1} \qquad c(\xi,\xi') = [4 + (\xi+\xi')^2]^{-1}$$
$$f(\xi,\xi') = 2(3+\eta)(1+\eta) - 2\xi\xi' - \xi(\xi+\xi')(1+\eta) - \xi'(\xi+\xi')(3+\eta) \tag{C.5}$$
$$g(\xi,\xi') = (3+\eta)(1+\eta) - \xi^2 - \xi(\xi+\xi')(2+\eta).$$

(C.3) can be normalized with the help of formula (35) of the text and it is not difficult to check that the resulting correlation function reduces correctly to the normalized fourth moment (36) when $\xi = \xi'$ as expected.

Appendix D

We require the inverse Fourier transform of equation (23) containing two products of Bessel functions. It is easy to show (from the power series representations) that (23) is an even function of k. Thus, ony the cosine transform is required.

Our starting point is equation 6.672.4 from [35]

$$\int_0^\infty K_v(\alpha x) I_v(\beta x)\cos(cx)\,\mathrm{d}x = \frac{1}{2\sqrt{\alpha\beta}} Q_{v-\frac{1}{2}}\left(\frac{\alpha^2+\beta^2+c^2}{2\alpha\beta}\right) \tag{D.1}$$

using x as the transform variable. This is valid for $\mathrm{Re}\,(\alpha) > |\mathrm{Re}\,(\beta)|$, $c > 0$, $\mathrm{Re}\,(v) > -\frac{1}{2}$. Here I and K are modified Bessel functions of the first and second kind, and Q is the Legendre function of the second kind. The modified Bessel functions can be written in terms of ordinary Bessel functions with imaginary arguments. We make the following substitutions: $\alpha = \varepsilon - \mathrm{i}a$, $\beta = -\mathrm{i}b$, a, b are real and positive, and ε is a small positive contant—we will eventually consider the limit as $\varepsilon \to 0$. We then have the following relations (using [35] 8.405.1, 8.406.1, and 8.407.2)

$$I_v(\mathrm{i}bx) - \mathrm{e}^{\mathrm{i}\pi v} J_v(bx)$$
$$K_v([\varepsilon - \mathrm{i}a]x) = \frac{\mathrm{i}\pi}{2}\mathrm{e}^{-\frac{\mathrm{i}\pi v}{2}}(J_{-v}([a+\mathrm{i}\varepsilon)]x) + \mathrm{i}Y_{-v}([a+\mathrm{i}\varepsilon]x)).$$

Substituting into (D.1) and neglecting terms in ε^2, gives

$$\mathrm{i}\int_0^\infty J_v(bx)J_{-v}([a+\mathrm{i}\varepsilon)x]\cos(cx)\,\mathrm{d}x$$
$$-\int_0^\infty J_v(bx)Y_{-v}([a+\mathrm{i}\varepsilon]x)\cos(cx)\,\mathrm{d}x = \frac{\mathrm{i}\mathrm{e}^{\mathrm{i}\pi v}}{\sqrt{ab}} Q_{v-\frac{1}{2}}(z) \tag{D.2}$$

where

$$z = \frac{a^2+b^2-c^2}{2ab} + \mathrm{i}\varepsilon\frac{a^2-b^2+c^2}{2a^3b}.$$

As we take $\varepsilon \to 0$, we need to consider three different regimes, depending on the value of z. We will assume that $a > b$ and v is real.

The first regime is $z > 1$. As $\varepsilon \to 0$ the Legendre function on the RHS of (D.2) becomes real, as do the two integrals on the LHS. Thus, equating real and imaginary parts on each side, we obtain

$$\int_0^\infty J_v(bx)J_{-v}(ax)\cos(cx)\,\mathrm{d}x = \frac{\cos(\pi v)}{\pi\sqrt{ab}} Q_{v-\frac{1}{2}}(z) \tag{D.3}$$
$$\int_0^\infty J_v(bx)Y_{-v}(ax)\cos(cx)\,\mathrm{d}x = \frac{\sin(\pi v)}{\pi\sqrt{ab}} Q_{v-\frac{1}{2}}(z). \tag{D.4}$$

The second regime is $-1 < z < 1$. Here the Legendre function has a branch cut along the real axis. As $\varepsilon \to 0$ the branch cut is approached from the upper half of the complex plane (because $a > b$). The value of the Legendre function then goes to ([35] 8.831.1 and 8.834.1)

$$Q_{v-\frac{1}{2}}(z) - \frac{\pi}{2\cos(\pi v)}(P_{v-\frac{1}{2}}(-z) - \sin(\pi v)P_{v-\frac{1}{2}}(z)) - \frac{\mathrm{i}\pi}{2}P_{v-\frac{1}{2}}(z).$$

Then, equating real and imaginary parts as before, we find that

$$\int_0^\infty J_v(bx)J_{-v}(ax)\cos(cx)\mathrm{d}x = \frac{1}{2\sqrt{ab}}P_{v-\frac{1}{2}}(-z) \tag{D.5}$$

$$\int_0^\infty J_v(bx)Y_{-v}(ax)\cos(cx)\mathrm{d}x = \frac{1}{2\cos(\pi v)\sqrt{ab}}(\sin(\pi v)P_{v-\frac{1}{2}}(-z) - P_{v-\frac{1}{2}}(z)). \tag{D.6}$$

Finally, for $z < -1$ we have ([35] 8.833.4)

$$Q_{v-\frac{1}{2}}(z) = -\mathrm{e}^{\mathrm{i}\pi(v-\frac{1}{2})}Q_{v-\frac{1}{2}}(-z)$$

which gives us

$$\int_0^\infty J_v(bx)J_{-v}(ax)\cos(cx)\mathrm{d}x = 0 \tag{D.7}$$

$$\int_0^\infty J_v(bx)Y_{-v}(ax)\cos(cx)\mathrm{d}x = \frac{1}{\pi\sqrt{ab}}Q_{v-\frac{1}{2}}(-z). \tag{D.8}$$

Using the relation

$$Y_v = \frac{\cos(\pi v)J_v - J_{-v}}{\sin(\pi v)} \tag{D.9}$$

one can use (D.3)–(D.8) to derive similar results for the transforms of other products of Js and Ys.

The result we require, i.e. the transfrom of (23), can be readily found by using (D.9) to write (23) in terms of $J_v J_{-v}$ products and applying the following two results ([35] 8.820.7 and 8.835.1)

$$P_v(z) = P_{-v-1}(z)$$
$$Q_v(z) - Q_{-v-1}(z) = \frac{\pi}{\tan(\pi v)}P_v(z).$$

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